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MEASURE OF INTERACTION IN A SYSTEM

AND

ITS APPLICATION TO CONTROL PROBLEMS

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FOREWARD

The Systems Research Center is a research and graduate study center operating in direct cooperation with all departments and divisions of Case Institute of Technology. The center brings together faculty and students in a coordinated program of research and education in the important techniques of systems theory, development, and application.

Research leading to this report was carried on by Dr. Mihajlo Mesarović, Visiting Professor of Engineering at Case Institute of Technology. Dr. Mesarović is Director of the Adaptive and Self-Organizing Systems group at the Systems Research Center. The research activities of this group was supported in part by the United States Office of Naval Research (Contracts NONR 1141 (09), NONR 1141 (12)) and the Ford Foundation.

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ABSTRACT

To enable a quantitative approach to the problems of control and synthesis of multivariable system a measure of interaction strength is proposed. Measure is defined as a property of the system under consideration in terms of the two conceptual experiments. Necessary and sufficient conditions for unit interaction are given for different types of linear systems. A statistical approach to the interaction measure is also discussed.

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MEASURE OF INTERACTION IN A SYSTEM AND ITS APPLICATION TO CONTROL PROBLEMS

I. Introduction

Problems of analysis and synthesis of a complex and multivariable systems present on one hand a special challenge to the science today in many different facets and on the other hand present a daily problem in many practical engineering situations; a problem which cannot be avoided or ignored any more. What is necessary, to cope with this situation, is a consistent approach to multivariable systems which will take into account the very specific nature of such systems. This viewpoint was discussed in detail in[1] where it was shown that in order to build such an approach it is necessary to attack the problems of interactions on a scientific, ergo, quantitative basis. This paper presents an initial attempt toward developing such an approach to the problem of interaction. It presents, in essence, concretization of the ideas developed in[1]. In addition to the discussion of the basic properties of interaction such as instances of unit and zero interactions a possible measure of the interaction strength is proposed.

It is interesting to review briefly how the problem of interaction has been treated so far in the control theory. In the early "servomechanism" age of the control theory the trend was to disregard the interactions and to treat every individual input-output pair by single variable techniques. Attempts to justify this assumption in practice by intentionally decoupling the multivariable system have almost never been successful. In the later "computer control" age of control theory a trend toward the other extreme has been developed. A complete knowledge of the process is assumed and the computer is assumed to generate the best possible manipulated input set to the system under control. Using state variable approach multi-dimensional and multivariable problems have been equated. The underlying philosophy was that the computer is or can be made to be almighty and therefore can answer all the problems no matter how complex is the system. It was not long before the pittfalls of such an approach were recognized.

In the present state of knowledge it might be rather safely assumed that an integrated, centralized, dynamically optimal computer control is on the verge of the impossible if the system is of a reasonable size, say with twenty state variables.

To cope with these problems a novel approach to control of complex multivariable systems has been suggested termed the organizational or multi-level goal-seeking approach.[2] Highlights of this approach are: reticulation of the system into the subsystem by grouping the strongest interacting variables; control every one of the subsystems in the best possible fashion using interacting control approach[3]; recompensating for the overlooked subsystems interactions by the coordinating action of a set of optimizers or controllers arranged in a hierarchy on different levels.

To implement such an approach to control of complex systems a measure of interaction is necessary. It is within the immediate framework of this approach that the subsequent development of the interaction measure took place.

It is of interest to notice that the multi-level goal-seeking approach is applied in many complex natural systems such as in the coordinating action of the central nervous system or in man-made systems such as industrial organizations[4]. One would expect, therefore, that similar principals might be useful when the complexity of the technical system under control becomes comparable to the complexity of the mentioned natural systems.

II. Basic Considerations of the Interaction Measure Concept

The concept of interaction introduced in this paper is based on the following considerations:

1. Interactions are viewed in a behavioristic fashion on the basis of the mutual influence of the two variables or two sets of variables regarding them as time functions and disregarding many non-abstract attributes of the system such as energy, spatial and temporal closeness, etc.

2. Interactions under consideration in this paper relate output variables.

In general interactions between the two subsystems can be approached in two ways:

a) "White box" approach - when one has access to the system and can recognize the subsystems and measure the terminal variables of the subsystems which are the inside variables of the system. The subsystem interaction is defined in terms of these inside variables.

b) "Black box" approach - when the observer does not have the access to the inside of the system. Interaction between the subsystems should be concluded upon from the observation of the system. The only subsystems which can be considered are those for which at least a subset of their terminal variables are at the same time terminal variables of the system. Two further distinctions can be made.

α) Cross-transfer interactions specify the relations among the sets of input and output variables.

β) Output interactions specify the relation among the subsets of the outputs set.

We shall consider in this paper β-category of interactions i.e. output interactions. The reasons which causes the emphasis on this type of interaction are the following: In a control situation, one is concerned with influencing the behavior of the output variables of the system in a best optimal way with the specified restrictions and constraints. The primary task in facilitating a multivariable control problem by reticulation is to subdivide the set of output variables into the subsets which will be influenced optimally (under constraints) in a spacially or temporally separated arrangements. Of special concern is how the control of one output subset i.e. of one sub-system, influences the behavior of the rest of the system ergo the remaining output subset. One would desire to group all the strongly interacting outputs into given groups and define the subsystems borders along the lines of weak outputs interaction.

3. Interactions are considered dynamically. They relate the time functions in a finite or infinite time intervals.

We are now in a position to define the black box output interactions as a dynamical property of an abstract or physical system. In order for a system attribute to be general it should be defined on the basis of the behavior of systems itself therefore on the basis of a conceptual "experiment" rather than on the basis of any specific analytical description of the system. To specify output interaction one has to observe the systems behavior in two different modes, therefore to perform two "experiments":

a) Observe the behavior (make the record of the time functions) of the two outputs "i" and "j" (two output sets) over the time interval under consideration.

b) Change the behavior (time function) of the output i and observe the change produced in the output j. The change produced now in output j will specify interaction K_{ji}.

Several remarks arise immediately:

1. Interaction K_{ji} is different than K_{ij}.
2. In general, especially for nonlinear system, interaction depends upon the operating conditions i.e. input and output sets in "experiment" α and also upon the change in output i - in experiment β.
3. Since outputs by definition cannot be changed directly, change of the output i - has to be produced by a change in the input sets. Interaction, therefore, depends upon this change too.

In view of these three remarks the experiments α and β should be carried out in the following way:

α) Change one input j (one input set) in a given way and also change simultaneously all remaining inputs in such a way that all the outputs except j-th remain unchanged.

$$\Delta y_j \neq 0 \; : \; \Delta y_k = 0 \quad k = 1,\ldots n \quad k \neq j \tag{1}$$

This modified experiment is necessary in order to recognize the input (sets) with respect to which the interaction is measured.

β) Keep the j-th input (set) unchanged and change all remaining inputs so as to produce a given change in i-th output and prevent changes in any output (set) other than i- and j-

$$\Delta x_j = 0 \; : \; \Delta y_k = 0 \quad k = 1,\ldots n \quad k \neq j, i \tag{2}$$

On the basis of the preceeding specification of the conditions for the two experiments three basic properties of interaction become apparent.

1. Interaction depends upon the outputs in both experiments and upon the specified input x_j

$$K_{ji}\left[y_j(t),\ y_j(t),\ y_i(t),\ y_i(t),\ x_j(t)\right] \qquad (3)$$

2. Interaction K_{ji} is zero if change in the i-th output (subset) produce no change in the j-th output (subset).

3. Interaction is of an infinite strength - termed unit interaction (because of the normalization introduced later) - if the changes in i-th and j-th outputs (subsets) are always mutually related functionally in the same way which relationship cannot be changed by any manipulation of the inputs. In other words some of the outputs completely specifies the others independent of the actual inputs. In the case of the analytically described systems one should be able to derive from the systems equations a relationship of the form

$$f\left[\bar{y}(t)\right] = 0 \ ; \ f \neq f\left[\bar{x}\right] \qquad (4)$$

in which input set does not appear explicitly. Note that the relationship f is neither linear nor static and can, in general, be a functional as well as a function. The dynamic characteristic of the interaction under consideration becomes apparent here.

Zero interaction can be easily recognized either from experiment or from the systems equations. Cases with unit interaction however presents special theoretical problems since the relationship (4) is not specified in any other way but solely by the absence of all the inputs.

In the following sections we shall give some theorems regarding unit interaction in linear systems. Although theorems in Sec. (IV) include theorems in (III) as special cases we shall follow inductive procedure to facilitate the understandings of the impact of the theorems.

III. Interactions in a System Subdivided into Two Subsystems

Consider a n-variable system with the input set $\bar{x}(t) = \{x_j(t)\}$ and the output set $\bar{y}(t) = \{y_j(t)\}$. Assume that the system is recognized as consisting of two subsystems S_1 and S_2 with the terminal variables $\overline{x^{(1)}}(t) = \{x_j^{(1)}(t)\}$; $\overline{y^{(1)}}(t) = \{y_j^{(1)}(t)\}$ and $\overline{x^{(2)}}(t) = \{x_j^{(2)}(t)\}$; $\overline{y^{(2)}}(t) = \{y_j^{(2)}(t)\}$ respectively. It is assumed here that input and output vectors of each sub-system are of the same order.

Interactions can be now defined between the two subsystems in terms of the both output vectors.

In experiment α) one is given an input vector $\bar{x}^{(1)}(t)$ while the changes of the output vector of the second subsystem $\bar{y}^{(2)}(t)$ should be compensated by the action of the input vector $\bar{x}^{(2)}(t)$. The conditions for performing experiment are therefore

$$\bar{x}^{(1)}(t) = \text{given} \qquad \bar{y}^{(2)}(t) = 0 \tag{5}$$

Since we are dealing here with the vector compensation the problems of existance and uniqueness of the required input set $\bar{x}^{(2)}(t)$ become of special importance[5]. Since the primary requirement is to eliminate any change which might occur in $\bar{y}^{(2)}(t)$ there exist a compensating set in the class of unbounded real functions if the system is linear. Uniqueness, however, of a linear multivariable system compensation depends upon the internal structure of the system. To make the outcome of the experiment α) nonambiguous one should specify additional constraints which together with condition (5) uniquely specifies the compensating vector $\bar{x}^{(2)}(t)$. This point is of a much greater importance for the three way subdivision of the system as will be shown in the next section.

Conditions for experiment β) are now

$$\overline{x^{(1)}}(t) = 0 \qquad \overline{y^{(2)}}(t) = \text{given} \tag{6}$$

Problems of existance now become prominent even for the linear system. If some of the elements of the output vector are not functionally independent due to the internal structure of the system under consideration, one cannot satisfy condition (6) for any $\bar{y}^{(2)}(t)$.

In what follows it will be assumed that the existance problems for the subsystem (S_2) have been considered in selecting $\bar{y}^{(2)}(t)$ and also that additional constraints have been specified to make the input vector unique.

The form of the conditions which specify unit interaction between the subsystems (S_1) and (S_2) depends upon the way in which the systems structure is represented. It is non-trivial to consider each of the representation separately since in the special case as for example for unit interaction it might not be possible to transform the system from one representation to another.

a) In state variable representation system is described now by the following vector equation

$$\frac{d\bar{z}}{dt} = \bar{A}\,\bar{z} + \bar{B}\,\bar{x} \tag{7}$$

$$\bar{y} = \bar{C}\,\bar{z}$$

where A, B and C are constant matrices, $\bar{z}$ is a state vector $\bar{y}$ is the output vector and $\bar{x}$ is the input vector. To simplify discussion we will assume that the state variables are defined as derivatives of the output variables so that the state variable vector can be partitioned in the following way

$$\bar{z} = \begin{Bmatrix} \bar{z}^{(1)} \\ \bar{z}^{(2)} \end{Bmatrix} \;:\; \bar{z}^{(1)} = \left\{ z_j^{(1)} \right\} \;:\; \overline{z^{(2)}} = \left\{ z_j^{(2)} \right\}$$

$$\overline{z_j^{(1)}} = \left\{ \frac{d^i y_j^{(1)}}{d\,t} \right\} \;:\; \overline{z_j^{(2)}} = \left\{ \frac{d^i y_j^{(2)}}{d\,t} \right\} \tag{8}$$

For a unit interaction in the system S_1 the following theorem holds

<u>Theorem 1</u>: A necessary condition for the system S_1 to have unit interaction is that the Jordan canonical form of matrix B has a zero-eigenvalue of multiplicity not smaller than the smallest order among the state vectors $\bar{z}_2^{(2)}$ and that the Jordan canonical form of the matrix $\bar{B}$ has multiplicative chain associated with this eigenvalue of the same order.

<u>Proof</u>: Using Jordan canonical form for $\bar{B}$ eq(7) can be written in the form

$$\bar{T}\left[\frac{d\bar{z}}{dt} - \bar{A}\,\bar{z}\right] = \bar{J}\,\bar{T}\,\bar{x} \tag{9}$$

where

$$\bar{B} = \bar{T}^{-1}\,\bar{J}\,\bar{T} \tag{10}$$

Under the conditions of theorem (1) the Jordan matrix J is of the form

$$\bar{J} = \begin{bmatrix} J_o & 0 & \cdots 0 \\ & J_1 & \\ 0 & & \ddots J_n \end{bmatrix} \tag{11}$$

where J_o is a k-th order matrix of the form

$$J_o = \begin{bmatrix} 0, & 1, & 0, & .. & 0 \\ 0, & 0, & 1, & .. & 0 \\ & & & .. & 1 \\ 0 & & ..0 & & 0 \end{bmatrix} \qquad (12)$$

The state vector can now be partitioned in the following way

$$\bar{z} = \begin{bmatrix} \bar{z}_p \\ \hline \bar{z}_{n-p} \end{bmatrix} \qquad (13)$$

where $\bar{z}$ is state vector of an output of p-th order.

Using the partition (13) of the state vector $\bar{z}$ one can write for Eq. (9)

$$\bar{P}_{p1} \left[\frac{d\bar{z}_p}{dt} - \bar{A}_{p1}\, \bar{z}_p - \bar{A}_{p2}\, \bar{z}_p \right] +$$

$$+ \bar{P}_{p2} \left[\frac{d\bar{z}_{n-p}}{dt} - \bar{A}_{n-p,1}\, \bar{z}_{n-p} - \bar{A}_{n-p,2}\, \bar{z}_{n-p} \right] = \bar{J}_o\, \bar{P}\, \bar{x} = 0 \qquad (14)$$

Since in Eq(14) no input appears the theorem is proven. Eq(14) also gives a relationship f which specifies the dynamics of the unit interaction.

Necessary conditions does not specify whether the unit interaction in the system is between the two subsystems or among the variables of any of the subsystems. This is specified by the equation (14) i.e. by the outputs which appear in it or rather by the structure of the matrices associated with this equation.

P - Canonical Representation

The system is represented now by the vector equation

$$\bar{Q} * \bar{y} = \bar{P} * \bar{x} \qquad (15)$$

where Q and P are linear differential operator matrices

$$\bar{Q} = \{q_{ij}\} \qquad \bar{P} = \{p_{ij}\}$$

and in addition $\bar{Q}$ is diagonal.

Both vectors $\bar{x}$ and $\bar{y}$ can be partitioned in terms of the terminal variables of the both subsystems

$$\bar{Q}_1 * \overline{y^{(1)}} = \bar{P}_{11} * \overline{x^{(1)}} + \bar{P}_{12} * \overline{x^{(2)}}$$

$$\bar{Q}_2 * \overline{y^{(2)}} = \bar{P}_{12} * \overline{x^{(1)}} + \bar{P}_{22} * \overline{x^{(2)}} \tag{16}$$

Unit interaction is now specified by the following theorems

__Theorem 2__: System (15) has a unit interaction if and only if the following condition is satisfied for any real valued vector function $\bar{u}(t)$

$$\left[\bar{P}_{11} * \bar{P}_{22} - \bar{P}_{12} * \bar{P}_{21}\right] * \bar{u}(t) = 0 \tag{17}$$

__Proof__: Using the properties of the linear operators the proof follows easily from Eq. (16). Applying $\bar{P}_{22}$ on the first equation and operator $\bar{P}_{12}$ on the second one obtains

$$\begin{aligned}
\bar{P}_{22} * \bar{Q}_1 * \overline{y^{(1)}}(t) &= \bar{P}_{22} * \bar{P}_{11} * \overline{x^{(1)}}(t) + \bar{P}_{22} * \bar{P}_{12} * \overline{x^{(1)}}(t) \\
\bar{P}_{12} * \bar{Q}_2 * \overline{y^{(2)}}(t) &= \bar{P}_{12} * \bar{P}_{21} * \overline{x^{(1)}}(t) + \bar{P}_{12} * \bar{P}_{22} * \overline{x^{(2)}}(t)
\end{aligned} \tag{18}$$

from where one obtains

$$\begin{aligned}
&\bar{P}_{22} * \bar{Q}_1 * \bar{y}^{(1)}(t) - \bar{P}_{12} * \bar{Q}_2 * \bar{y}^{(2)}(t) = \\
&\left[\bar{P}_{22} * \bar{P}_{11} - \bar{P}_{12} * \bar{P}_{21}\right] * \bar{x}^{(1)}(t)
\end{aligned} \tag{19}$$

Therefore condition (17) is sufficient. Since only two vector variables are involved to prove that condition (17) is necessary it is enough to exhibit the only other possible combination

$$\begin{aligned}
&\bar{P}_{21} * \bar{Q}_1 * \bar{y}^{(1)}(t) - \bar{P}_{11} * \bar{Q}_2 * \bar{y}^{(2)}(t) = \\
&\left[\bar{P}_{11} * \bar{P}_{22} - \bar{P}_{21} * \bar{P}_{12}\right] * \bar{x}^{(2)}(t)
\end{aligned} \tag{20}$$

which follows easily from (18). Since the linear operators can exchange the order in which they are applied the theorem is proved.

Condition (17) specifies unit interaction between the two subsystems only. If the unit interaction inside any of the subsystems i.e. among some of their terminal variables this appears in the form of the functional dependence among the elements of the output vectors of the given subsystem. This in turn provoked the existance and uniqueness problem in the α) and β) experiments but does not influence directly condition (17). Therefore condition (17) really specified only unit interaction between the subsystems.

c. <u>V-Canonical Representation</u>

the Systems vector equation is now

$$\bar{L} * \bar{y} = \bar{N} * \bar{x} + \bar{R} * \bar{y} \tag{21}$$

where $\bar{L}$, $\bar{N}$ and $\bar{R}$ are linear differential operator matrices, $L = (\ell_j)$ $N = (n_j)$; $R = (r_{ji})$ and L and N are diagonal while R is off diagonal i.e. $r_{ij} = 0$ for all $j = i$.

It is apparent that the system has unit interaction if and only if one of the following conditions is satisfied

1) $$\Delta [\bar{N}] * u(t) = 0 \tag{22}$$

for any real function u(t) i.e. if any of the diagonal elements of the matrix $\bar{N}$ is zero.

2) If $$|r_{ij}| \longrightarrow \infty \tag{23}$$

for one or more j-s.

IV. <u>Interactions in a System Subdivided into Three Subsystems</u>

If the system is subdivided into three subsystems S_1 S_2 and S_3 one can distinguish two situations with respect to interactions.

A) One can consider interactions between one subsystem say S_1 and the other two subsystems S_2 and S_3. The problem is then reduced to the one discussed in Section III.

B) One can consider interactions between the two subsystems say S_1 and S_2 which together with the subsystem S_3 composed the system. This interaction will be termed relative interaction. The problem is now conceptually different from that discussed in Section III. Interaction is considered between the two parts of the system S_1 and S_2 in the presence of another subsystem S_3. This presence conditioned interaction $K_{(2)(3)}$ although a direct influence of the subsystems S_3 should be eliminated. This can be done by securing as the primary requirement in the consideration

$$\overline{y^{(3)}}(t) = 0 \tag{24}$$

To achieve this as well as to secure other requirements of the experiments input vector $\overline{x^{(3)}}$ should be used so that in general

$$\overline{x^{(3)}}(t) \neq 0 \tag{25}$$

Conditions for experiments α and β regarding subsystems S_1 and S_2 are as before Eq (5) and (6). Additional requirements (24) and (25) however caused quite different behavior of the system as it will become evident from the following discussion.

a) State variable representation

Conditions for relative unit interaction are similar to the ones stated in Theorem 1. If the Jordan canonical form of the matrix $\bar{B}$ is of the required form Eq. (9) indicates that there is a unit relative interaction in the system. If in addition $\overline{y^{(3)}}$ does not appear in Eq. (9) and the homogeneous differential equation in $\overline{y^{(1)}}$ and $\overline{y^{(2)}}$ has non-trivial solutions unit relative interaction exists among some of the elements of the vectors $\overline{y^{(1)}}$ and $\overline{y^{(2)}}$.

b) P-Canonical Representation

This representation is very suitable to indicate the basic differences regarding relative interactions between the two subsystems in the presence or in the absence of another subsystem S_3.

The systems vector equation can now be partitioned in the following way:

$$\begin{aligned}
\bar{Q}_1 * \overline{y^{(1)}} &= \bar{P}_{11} * \overline{x^{(1)}} + \bar{P}_{12} * \overline{x^{(2)}} + \bar{P}_{13} * \overline{x^{(3)}} \\
\bar{Q}_2 * \overline{y^{(2)}} &= \bar{P}_{21} * \overline{x^{(1)}} + \bar{P}_{22} * \overline{x^{(2)}} + \bar{P}_{23} * \overline{x^{(3)}} \\
\bar{Q} * \overline{y^{(3)}} &= \bar{P}_{31} * \overline{x^{(1)}} + \bar{P}_{32} * \overline{x^{(2)}} + \bar{P}_{33} * \overline{x^{(3)}}
\end{aligned} \tag{24}$$

Conditions for relative unit interaction can be now specified by the following theorem

Lemma: 1. If the following condition is satisfied for every real function $\overline{u(t)}$

$$\left| \bar{P}_{33} * \bar{P}_{12} - \bar{P}_{13} * \bar{P}_{32} \right| * u(t) = 0 \tag{25}$$

the relative interaction between the system S_1 and S_2, is zero.

2. If the following condition is satisfied for every real function $\overline{u(t)}$

$$\left| \bar{P}_{33} * \bar{P}_{22} - \bar{P}_{13} * \bar{P}_{32} \right| * u(t) = 0 \tag{26}$$

the relative interaction between the subsystems S_1 and S_2 is undeterminable by the experiments α and β.

Theorem: A system subdivided into three subsystems has a unit relative interaction among the subsystems S_1 and S_2 if and only if: 1) the following condition is satisfied for every real time function u(t).

$$|\bar{P}| * u(t) = 0 \tag{27}$$

where $|\bar{P}|$ is determinant of the operators from Eq(24); 2) if the system does not have zero relative interaction specified by the lemma 1 or indeterminate relative interaction specified by lemma 2.

Proof: Consider at first experiment α. The conditions for the experiment are

$$\begin{aligned} \bar{Q}_2 * \overline{y^{(2)}} &= \bar{P}_{21} * \overline{x^{(1)}} + \bar{P}_{22} * \overline{x^{(2)}} + \bar{P}_{23} * \overline{x^{(3)}} = 0 \\ \bar{Q}_3 * \overline{y^{(3)}} &= \bar{P}_{31} * \overline{x^{(1)}} + \bar{P}_{32} * \overline{x^{(2)}} + \bar{P}_{33} * \overline{x^{(3)}} = 0 \end{aligned} \tag{28}$$

Using multiplicative properties of linear operators one obtains from (24) and (28)

$$\left[\bar{P}_{22} * \bar{P}_{33} - \bar{P}_{23} * \bar{P}_{32} \right] * \bar{Q} * \overline{y^{(1)}} = |\bar{P}| * \overline{x^{(1)}} \tag{29}$$

From the right side of Eq (29) one can conclude that condition 1 of the theorem is sufficient. From the left side one can conclude that the condition of the lemma 1, causes undeterminancy in the outcome of the experiment α).

Conditions for the experiment β) give the equations

$$\bar{Q}_3 * \overline{y^{(3)}} = \bar{P}_{32} * \overline{x^{(2)}} + \bar{P}_{33} * \overline{x^{(3)}} = 0$$

$$\bar{Q}_1 * \overline{y^{(1)}} = \bar{P}_{12} * \overline{x^{(2)}} + \bar{P}_{13} * \overline{x^{(3)}} \qquad (30)$$

$$\bar{Q}_2 * \overline{y^{(2)}} = \bar{P}_{22} * \overline{x^{(2)}} + \bar{P}_{23} * \overline{x^{(3)}}$$

By elimination one can obtain from these

$$\bar{P}_{33} * \bar{Q}_1 * \overline{y^{(1)}} = \left[\bar{P}_{33} * \bar{P}_{12} - \bar{P}_{32} * \bar{P}_{13}\right] * \overline{x^{(2)}}$$

$$\bar{P}_{33} * \bar{Q}_2 * \overline{y^{(2)}} = \left[\bar{P}_{33} * \bar{P}_{22} - \bar{P}_{32} * \bar{P}_{23}\right] * \overline{x^{(2)}} \qquad (31)$$

From Eq (31) follows that the $\overline{y^{(1)}} = 0$ under the condition of the lemma regardless of the $y^{(2)}$ so that the relative interaction between S_1 and S_2 is zero. Since Eqs. (29) and (31) give complete information about the outcome of the both experiments the condition 1 of the theorem is also necessary for unit relative interaction.

It is interesting to note that lemma 1 specifies a zero relative interaction between the subsystems S_1 and S_2 only because of the primary requirements that $\overline{y^{(3)}} = 0$. Really from Eq. (31) one obtains

$$\bar{V} * \overline{y^{(2)}} = \bar{\bar{x}} * \overline{y^{(3)}} \qquad (32)$$

where $\bar{V}$ and $\bar{\bar{W}}$ are linear matrix operators. Eq. (32) shows that the relative interaction among S_2 and S_3 is a unit one. Therefore zero relative interaction between S_1 and S_2 indicates only that a much stronger interaction exists between S_1 and S_3 (infinite interaction). Actually if the interaction between S_1 on one side and S_2, S_3 on the other is considered (Case A) this interaction is not necessarily unit one in spite of the condition (25).

C) V-Canonical Representation

Equations for the system are now

$$\bar{L}_1 * \overline{y^{(1)}} = \bar{N}_1 * \overline{x^{(1)}} + \bar{R}_{12} * \overline{y^{(2)}} + \bar{R}_{13} * \overline{y^{(3)}} \qquad (33)$$

$$\bar{L}_2 * \overline{y^{(2)}} = \bar{N}_2 * \overline{x^{(2)}} + \bar{R}_{21} * \overline{y^{(1)}} + \bar{R}_{23} * \overline{y^{(3)}}$$

$$\bar{L}_3 * \overline{y^{(3)}} = \bar{N}_3 * \overline{x^{(3)}} + \bar{R}_{31} * \overline{y^{(1)}} + \bar{R}_{32} * \overline{y^{(2)}} \quad (33)$$

Experiment β results for the relation

$$\bar{L}_1 * \overline{y^{(1)}} = \bar{N}_1 * \overline{x^{(1)}} \quad (34)$$

while experiment β gives

$$\bar{L}_1 * \overline{y^{(1)}} = \bar{R}_{12} * \overline{y^{(2)}} \quad (35)$$

It is apparent that condition for unit interaction is again

$$\Delta \mid \bar{N}_1 \mid = 0 \quad \text{or} \quad \| R_{12} \| \longrightarrow \infty \quad (36)$$

V. A Measure of Interaction

Considerations in the preceeding sections enables characterization of zero and unit interaction in a system. The main problem left, however, is how to characterize interactions in the cases other than zero and unit. The answer to this question is very difficult. To characterize these cases it is necessary to introduce a measure of interaction which will serve as a basis for comparison of the two interactions of the same system or to compare different systems regarding the interaction.

Before proposing a measure of interaction consider the intuitive requirements which such a measure should satisfy:

1. It should be a functional k_{ji} defined in the set of the terminal variables.
2. It should be equal to zero if and only if the interaction is zero.
3. It should be equal to one if and only if the system has an unit interaction.
4. It should be a single-valued function of the systems parameters or characteristic functions with the domain between zero and one.

5. It should be monotone between any consecutive intersections with the lines $k_{ji} = 0$ and $k_{ji} = 1$.

A measure of interaction can be obtained in the following way.

Define a functional of the output time function $\phi\left[y_j\right]$ which satisfies requirements 1.-5. except that the domain of the functional is from zero to infinity. Take the ratio of the two functionals obtained from the experiments β and α

$$K_{ji} = \frac{\phi\left[\Delta_\beta y_j\right]}{\phi\left[\Delta_\alpha y_j\right]} \tag{37}$$

The interaction measure is now

$$k_{ji} = \frac{K_{ji}}{1 + K_{ji}} \tag{38}$$

It is apparent that the interaction measure as defined in (37) is not unique. It depends on the selection of the functional ϕ which in turn should reflect the context in which the measure is used and also should be as simple as possible. It depends also on the test signal used. The importance of the imput amplitudes, however, can be eliminated if the following limit exists

$$\lim K_{ji} = \lim_{x_j \to 0} \left[\frac{\phi\left[\Delta_\beta y_j\right]}{\phi\left[\Delta_\alpha y_j\right]}\right] \tag{39}$$

In the case of the vector subsystems interaction measure (37) becomes

$$K_{(j)(i)} = \sum_{e=1}^{p_j} \left[\frac{\phi\left[\Delta_\beta y_e^{(j)}(t)\right]}{\phi\left[\Delta_\alpha y_e^{(j)}(t)\right]}\right] \tag{40}$$

For a linear system a simple mean square functional may be appropriate

$$\phi\left[\Delta y\right] = \int_T \left[\Delta y(t)\right]^2 dt \tag{41}$$

where T is time interval in which the systems behavior is under consideration.

For a final value system functional might be defined in terms of deviation of the final state

$$\phi = [\Delta y(T)]^2 \tag{42}$$

For a linear system and V-canonical representation one obtains

$$K_{ji} = \frac{\int_T [R_{ji} * \Delta y_i(t)]^2 \, dt}{\int_T [N_j * \Delta x_j(t)]^2 \, dt} \tag{43}$$

Assume the following selection for the test change $y_i(t)$ is made

$$y_i(t) = \Delta x_j(t) = a\Delta z_j(t) \tag{44}$$

Since limit (39) now exists

$$K_{ji} = \lim_{a\Delta \to 0} \frac{\int_T [R_{ji} * a\Delta z_j(t)]^2 \, dt}{\int_T [N_j * a\Delta z_j(t)]^2 \, dt} =$$

$$\frac{\int_T [R_{ji} * \Delta z_j(t)]^2 \, dt}{\int_T [N_j * \Delta z_j(t)]^2 \, dt} \tag{45}$$

measure is independent of the inputs amplitudes.

It is interesting to note that the measure is defined very simply in terms of the V-canonical representation. This seems to be true even for the nonlinear systems. Therefore V-canonical representation appears to be most appropriate.

VI. A Statistical Approach to the Interaction Measure

A measure of interaction as defined in (37) depends upon the form and in general even upon the amplitudes of the input vector. This pittfall can be reduced if the measure is defined statistically on a whole set of input vectors.

Assuming that the set is finite one obtains for the measure

$$K_{ji} = \frac{M\left\{\sum_{q}^{p} \phi\left[\Delta_\beta y_j(t\ ,\ q)\right]\right\}}{M\left\{\sum_{q}^{p} \phi\left[\Delta_\alpha\ y_j(t\ ,\ q)\right]\right\}} \tag{46}$$

where M indicates operation of mathematical expectation and p is the dimension of the set. If the set is infinite and can be specified with a parameter λ one obtains

$$K_{ji} = \frac{M\left\{\int_\Omega \phi\left[\Delta_\beta y_j(t\ ,\lambda\ )\right]\ d\lambda\ \right\}}{M\ \left\{\int_\Omega \phi\left[\Delta_\alpha y_j(t\ ,\lambda\ )\right]\ d\lambda\ \right\}} \tag{47}$$

Consider several examples of the input sets. Assume that the input set is a real vector space generated by a basis $\left[z_1(t),\ \ldots\ z_n(t)\right]$. Interaction measure becomes now

$$K_{ji} = \frac{\sum_q\left[M\left\{V_q^{\ 2}\right\} \int_T\ \Delta_\beta^{\ 2}\ y(t\ ,\ q)\ dt\right]\Big\}}{\sum_q\left\{M\left\{V_q\ \right\} \int_T \Delta_\alpha^{\ 2}\ y\ (t\ ,\ q)\ dt\ \right\}} \tag{48}$$

If the vector space Ω_x is of infinite dimensions

$$x(t) = \int_S V(s)\ z(s,\ t)ds$$

interaction measures become

$$K_{ji} = \frac{\int_S\left\{\left[\int_T \Delta_\beta^{\ 2}\ y_j(t\ ,\ s)\ dt\right]\ M\ \left[V_s^{\ 2}\right]\right\}\ ds}{\int_S\left\{\left[\int_T \Delta_\alpha^{\ 2}\ y_j(t\ ,\ s)\ dt\right]\ M\ \left[V_s^{\ 2}\right]\right\}\ ds} \tag{49}$$

Preceeding discussion indicates the structure of the proposed measure of interaction. Many problems of the application of this measure, however, are outside of the scope of this paper.

VII. Behavior of the System in the Constrained Environment - Cohesion of the Systems

Interaction is considered to be the property of a system which should be as much as possible independent of the circumstances in which the system operates. If the behavior of the system is restricted, however, this has to show up in the measure of interaction and in fact in analytical or experimental determination of the measure. To take into account these restrictions the concept of cohesion has been introduced. Its definition is based on the following: Consider a system with the given constraints reflecting physical realization or economical optimum. There is a given set of output vectors which now can be achieved without any deviation or error by acting on the system with a set of inputs from the given restricted class. With respect to these outputs the system will be defined to have zero cohesion. Outputs outside the class of zero cohesion can be achieved by the given system only approximately. The success in the achieving these outputs define cohesion of the system in the given constrained conditions. It is interesting to note that if the desired outputs behavior is specified only by a point in the state-variable hyperspace independent of the time when this point is reached one has a special problem in cohesion which under the name of controlability has been studied by several authors.

VIII. Application of the Interaction Measure in Control

Measure of interaction should become standard concept in the analysis and synthesis of the large control system. The areas of its application might be divided into three parts.

1. In analyzing large complex systems one should establish matrix of relative output interaction $\bar{K} = \{k_{ij}\}$ which will serve as a basis for determining which approach will be taken in solving control problems. For example, the interaction matrix will indicate whether some of the subsystems can be treated independently and also whether interactions inside a given subsystem are strong or weak. For a strongly interacting system a centralized control can be applied while for the weakly interacting systems cross-controller approach might be more appropriate.

2. Interaction measure offers a method for realization of organizational approach discussed in the Introduction. The system can now be subdivided into more subsystems than in case 1 since the introduction among the subsystems will be taken care of by the controllers on the higher level.

3. Interaction in a system can be evaluated continuously and used in an adaptive interacting control approach.

IX. Conclusions

The present paper offers a conceptual basis for a quantitative approach to the problem of interaction and also gives an analytical basis as to how the problem can be solved for linear systems. The proposed approach however opens a vast area of problems, some of them as e.g. those connected with non-linear systems are of great complexity. The problems which need immediate further study, however, are those connected with the application of the approach. In this respect it is of particular significance that the measure of interaction is defined for "black box" systems, and, therefore, the associated problems can be studied without use of involved analytical apparatus but rather by using a computer or experiments on actual systems.

X. References

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